Eyes on $e$

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Outline

I. Natural number $e = 2.7183\ldots$ (Euler)

II. Complex numbers and Euler’s identity

III. Ubiquity of $e$ and $\pi$

IV. Conclusions
Terminal velocity

Terminal velocity sphere in oil and water

Boris V Balakin, Alex C Hoffmann, Pawel Kosinski & Lee D Rhyne (2010)

\[ V(t) = V_0 \left( 1 - e^{-t/\tau} \right) \]
\[ e = 2.718281828... \]

\( \tau \): time scale, set by viscosity. Relatively small for sphere dropped in oil; larger for sphere dropped in water
Leonhard Euler (1727)

“Meditatio in Experimenta explosione tormentorum nuper instituta”

(Meditation upon experiments made recently on the firing of Canon)

Swiss, 1707-1783, student of Johann Bernouilli
Compound interest problem in a savings account

Initial deposit

Final balance

100% interest in a one time payout
Interest payout scheme:

once  twice  daily
Payout scheme of 100%:

once         twice           daily

2000W         2250W           2718W
# Account Balance over time

## Payout scheme:
- once
- twice
- yearly

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<th>2018</th>
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# Account Balance over time

<table>
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<tr>
<th>Year</th>
<th>Once</th>
<th>Twice</th>
<th>Yearly</th>
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<td>2000</td>
<td>2250</td>
<td>2674</td>
<td>2718</td>
</tr>
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</table>

_**Payout scheme:**_
- daily
- once
- twice
- yearly
Payout over n periods ("installments")

\[
1 + \text{gain} = \frac{\text{final deposit}}{\text{initial deposit}} = \frac{2718}{1000} = 2.718
\]

daily, \( n = 10950 \) in 30 yr

\[
1 + \text{gain} = \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)\ldots \rightarrow e = 2.718281828\ldots
\]

(n times)

large \( n \) limit
Account balance by daily compound interest

2.71828... x 1000
Definition of exponential function

Define
\[ e^x = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n \]

Can show
\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \ldots = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

\[
1! = 1 \\
2! = 2 \\
3! = 6 \\
4! = 24 \\
n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n
\]

\[
1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.7167 \quad (e^1 = 2.71828)
\]

\[
1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \frac{32}{5!} = 7.2667 \quad (e^2 = 7.3891)
\]

Example

\[ n! \approx \sqrt{2\pi n} \frac{n^n}{e^n} \] (Stirling formula)

n! grows much faster than \( x^n \), so that the series converges for all x
Alternate definition: slope(x) = exp(x)

\[
\begin{cases}
    \frac{d}{dx} f(x) = f(x) \\
    f(0) = 1
\end{cases}
\]

Unique solution: \( f(x) = e^x \)

Exercise: show key property \( f^n(x) = f(nx) \)
II. Complex numbers and Euler’s identity
Leonhard Euler’s complex numbers (student of Johann Bernouilli)

(1707-1783)
The imaginary number $i$

$i^2 = -1,$

$i = \sqrt{-1}$

$2\alpha = \pi$

$\alpha = \frac{\pi}{2}$
Complex numbers $z$

$z$ with length 2 and angle $\frac{1}{3} \pi$
Square of a complex number

\[ w = z^2 \text{ described by} \]
\[ |w| = |z|^2 \]
\[ \beta = 2\alpha \]

\[ w = z^2 \text{ has length 4 and angle } \frac{2}{3}\pi \quad \text{←} \quad |z| = 2, \, \alpha = \frac{1}{3}\pi \]
Pythagoras on the unit circle

\[ x^2 + y^2 = 1 \]
\[ \cos^2 \alpha + \sin^2 \alpha \equiv 1 \]

\[ z = x + iy = \cos \alpha + i \sin \alpha \]
Powers on the unit circle $S^1$

$|z| = 1 \rightarrow w = z^n$

$|w| = 1, \beta = n\alpha$

$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha$

De Moivre
Powers of the exponential function

\[ e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \]

\[ (e^x)^n = e^{nx} \]

\[ e^{ix} = 1 + ix + \frac{1}{2}(ix)^2 + \frac{1}{6}(ix)^3 + = 1 + ix - \frac{1}{2}x^2 - \frac{1}{6}ix^3 + \]

\[ e^{ix} = \left(1 - \frac{1}{2}x^2 + \right) + i \left(x - \frac{1}{6}x^3 + \right) \]

\[ (e^{ix})^n = e^{nix} \]
Powers on $S^1$ \[(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha\]

Powers of $e^{ix}$ \[\left(e^{ix}\right)^n = e^{nix}\]

$x = \alpha$:

\[e^{i\alpha} = \cos \alpha + i \sin \alpha\]

\[\left(e^{i\alpha}\right)^n = e^{i n\alpha} = \cos n\alpha + i \sin n\alpha\]

\[e^{i\alpha} = \cos \alpha + i \sin \alpha:\]

\[\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} = \Re e^{i\alpha}\]

\[\sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \Im e^{i\alpha}\]
Polar representation of complex numbers

\[ z = r (\cos \alpha + i \sin \alpha) = r e^{i\alpha} \]

\[ z^n = r^n e^{i n \alpha} \]

\[ \cos \alpha + i \sin \alpha = \cos (\alpha + 2\pi n) + i \sin (\alpha + 2\pi n) (n \in \mathbb{N}) \]

\[ e^{i\alpha} = e^{i(\alpha + 2\pi n)} (n \in \mathbb{N}) \]

\[ 1 = e^{i2\pi n} (n \in \mathbb{Z}) \]

\[ \log z = \log r + i\alpha + i2\pi n (n \in \mathbb{N}) \]
What have we learned

e and $\pi$ are associated by Euler’s identity

\[ e^{i0} = 1, \quad e^{\frac{1}{2}\pi i} = i, \quad e^{i\pi} = -1, \quad e^{\frac{3}{2}\pi i} = -i, \quad e^{2\pi i} = 1, \quad \text{etc} \]
Euler’s identity

\[ e^{i\pi} + 1 = 0 \]

\( e = 2.71828 \ldots, \ pi = 3.14159 \)

“The most remarkable formula in mathematics” (Richard Feynman)

Example:

\[ 1 + \frac{i\pi}{1!} - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{i\pi^9}{9!} = -0.9760\ldots + 0.0069\ldots i \]
Q1. Complex numbers

Find the square $z^2$ of the number $z=1+i$.

What is the square of $z=1-i$?
What have we learned

Complex numbers:

a natural extension of the real numbers

exponential map $w = e^z$ in the complex plane

$w$ is oscillatory

$w$ decays exponentially

$w$ grows exponentially

$w$ is oscillatory

Im($z$)

Re($z$)
III. Ubiquity of e and pi
Fourier series

\[ f(x) = C_0 + C_1 e^{i\omega_0 x} + C_2 e^{2i\omega_0 x} + C_3 e^{3i\omega_0 x} + \cdots \]

- Low frequency, large amplitude
- High frequency, small amplitude
Fourier series of a periodic block function

\[ f(x) = \begin{cases} 0, & -a < x < 0 \\ A, & 0 < x < a \\ 0, & a < x < 2a \end{cases} \]

\[ C_n = A \frac{\sin nk_0 a}{nk_0 a}, \quad k_0 = \frac{2\pi}{L} \]
Heat flows from hot to cold
Fourier analysis of heat flow

$$u(t, x) : \begin{cases} 
u_t = u_{xx} \\ u(0, x) = u_0(x) \end{cases}$$

$$u = a(t)e^{ikx} :$$

$$\frac{da(t)}{dt} = -k^2a(t)$$

$$a(t) = a(0)e^{-k^2t}$$

Exponential decay in time for $k > 0$

Higher modes quickly die out, leaving only $k=0$ at large times
Q2. Heat flow

Consider a pair of metal bars of the same size and material, initially at $T_1=100$ and $T_2=200$. Upon bringing them into contact, what is their equilibrium temperature at late times?

[Hint: Heat is thermal energy, measured by temperature. Upon contact, heat flow sets in preserving total energy. The total energy at late time is the same as the total initial energy.]
Time harmonic motion

\[ u = A \cos \omega t \]

Angular frequency

\[ \omega = \frac{2\pi}{P} \]
Angular frequency and wave number

\[ \omega = \frac{2\pi}{P} \]

\[ k = \frac{2\pi}{\lambda} \]
Wave propagation

\[ u = A \cos(\omega t - kx) : \]

Wave equation

\[ u_{tt} - c^2 u_{xx} = (-\omega^2 + c^2 k^2)u = 0 \]

relation with velocity \( c \)

\[ \omega = ck \]

Linear dispersion relation (dispersionless medium)
Fourier analysis of wave motion

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \\
u(0,x) &= u_0(x) \\
u_t(0,x) &= u_1(x)
\end{align*}
\]

\[
u = a(t)e^{ikx} : \quad \frac{d^2 a(t)}{dt^2} = -k^2 a(t), \quad a(t) = a(0)e^{\pm ikt} \quad (u_1 = 0)
\]

Each Fourier mode propagates without decay: persistent wave motion

(shape may change, total energy and momentum are conserved)

Jean-Baptiste le Rond d’Alembert (1717-1983)
Q3. Sound waves

The linear dispersion relation for sound waves shows that angular frequency and wave number have a constant ratio.

In a duet, A sings at A4 (440 Hz), B sings at E5 (660 Hz). What are the ratios the wave lengths of their sounds?
de Broglie matter wave

Probability \[ |\psi(x)|^2 = a^2(x) \]

Wave function

\[ \psi(t,x) = a(x)e^{-i\omega t} \]

\[ E = \hbar \omega = i\hbar \frac{\partial}{\partial t} \phi \]

is eigenvalue in

\[ i\hbar \frac{\partial}{\partial t} \psi = E\psi \]
Schrodinger wave function

\[ i\hbar \frac{\partial}{\partial t} \psi = \hat{H} \psi \]

\[ p = mv : \quad E_k = \frac{1}{2}mv^2 = \frac{p^2}{2m} \]

\[ H = E_k = \frac{p^2}{2m} \rightarrow \hat{H} = \frac{\hat{p}^2}{2m}, \quad \hat{p} = i\hbar \partial_x \]

\[ -i\hbar \Psi_t = \frac{\hbar^2}{2m} \Psi_{xx} \]
Exponential function in relativity

Minkowski spacetime

$t = x$  
$(c = 1)$

light cone
Hyperbolic functions

\[ e^{i\alpha} = \cos \alpha + i \sin \alpha \]

\[ \cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} = \text{Re} e^{i\alpha} \]

\[ \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \text{Im} e^{i\alpha} \]

\[ \cos^2 \alpha + \sin^2 \alpha = 1 \]

\[ e^{\lambda} = \cosh \lambda + \sinh \lambda \]

\[ \cosh \lambda = \frac{e^{\lambda} + e^{-\lambda}}{2} = \cos i\lambda \]

\[ \sinh \lambda = \frac{e^{\lambda} - e^{-\lambda}}{2} = -i\sin i\lambda \]

\[ \cosh^2 \lambda - \sinh^2 \lambda = 1 \]
Hyperbolic representation of a world-line

\[ x^a = (t, x), \quad u^a = \left( \frac{dt}{ds}, \frac{dx}{ds} \right), \quad u^c u_c = u^t u^t - u^x u^x = 1 \]

\[ u^t = \cosh \lambda, \quad u^x = \sin \lambda \]

\[ V = \frac{dx}{dt} = \frac{dx}{ds} / \frac{ds}{dt} = \frac{u^t}{u^x} = \tanh \lambda \]
Rest mass energy

\[
\cosh \lambda = \sqrt{1 + \sinh^2 \lambda} \approx 1 + \frac{1}{2} \sinh^2 \lambda \approx 1 + \frac{1}{2} \tanh^2 \lambda \quad (\lambda \ll 1)
\]

\[p^a = mu^a \text{  relativistic four-momentum}\]

\[
E = mu^t = m \cosh \lambda = m \sqrt{1 + \sinh^2 \lambda}
\]

\[
P = mu^x = m \frac{dx}{d\tau}
\]

\[
E = \sqrt{E_0^2 + P^2} \approx m + \frac{1}{2} mV^2 = E_0 + E_k
\]

Restore velocity of light c: \[E_0 = mc^2\]
Rest mass energy

\[ E_0 = mc^2 \]

dispersion relation of light

dispersion relation of \( m \) (books, your laptop, ...)

relativistic x-momentum
Q4. Powering a laptop

Consider an new battery converting up to 1 g of mass into electrical power.

A laptop uses up to 100 W (Joules per second). For how many years can this battery power our laptop?
e^z epitomized by Euler’s identity $e^{i\pi}+1=0$.

e^z describes exponential growth and harmonic motion, in finance, heat flow, sound waves, quantum mechanics…

e^z describes our spacetime structure, of propagation of light