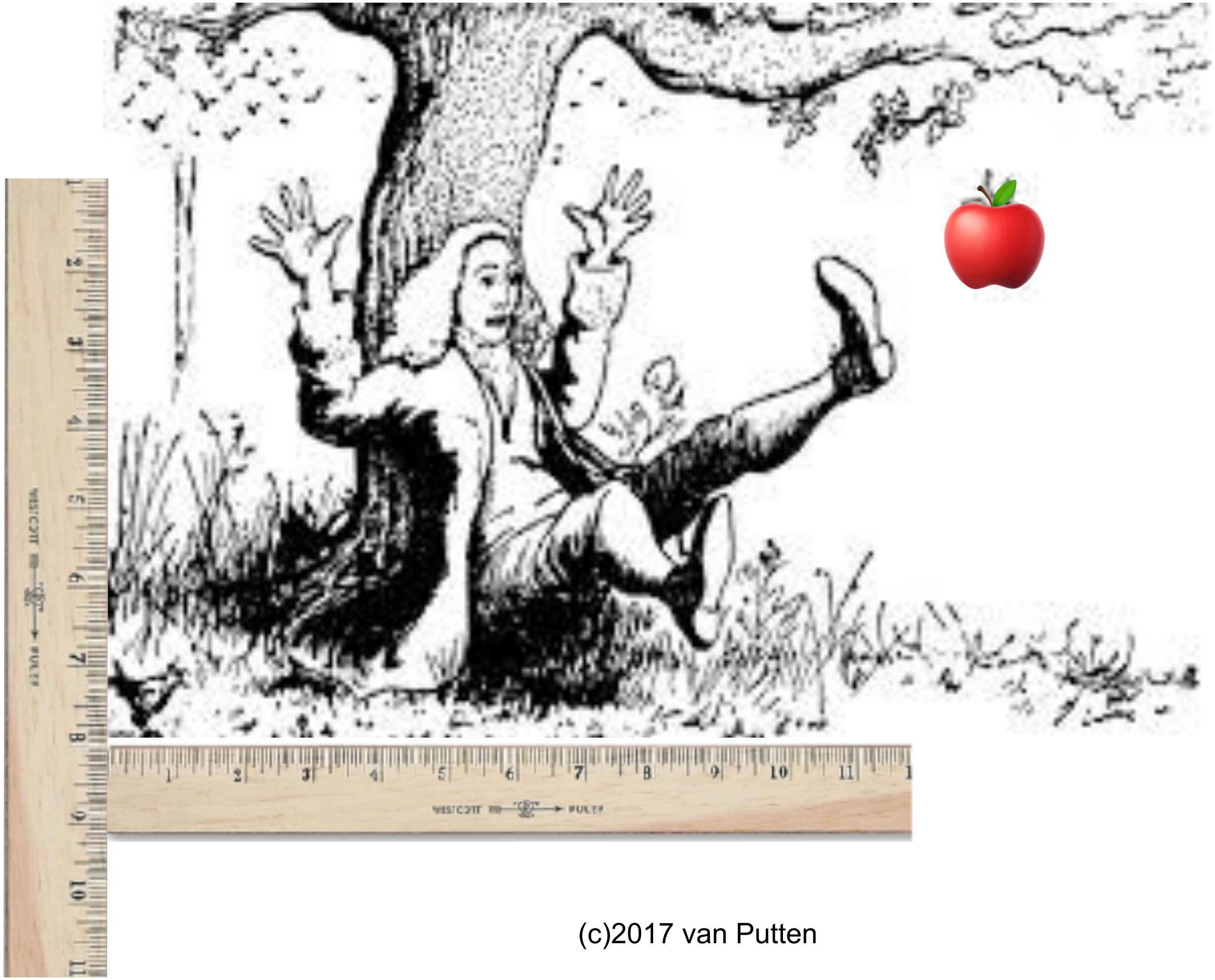
Elements of classical mechanics



Contents

II. (Last time) Open and closed: unbound and bound (*hyperbolic* and *elliptic*)

III. Three classical mechanics problems

Hooke's spring

Newton's apple

The jumper

Classification of orbits by H

$$\boxed{H = E_k + U = \frac{1}{2}mv^2 - \frac{GMm}{r}}$$

Since $E_k \geq 0, U \rightarrow 0$ upon approaching infinity

$H < 0$ **bound orbits:** *forbidden to reach infinity*

$H > 0$ **unbound orbits:** *allowed to reach infinity*

“Black objects”

1793: John Michell

1795: Pierre Laplace

1915: Karl Schwarzschild (exact solution)

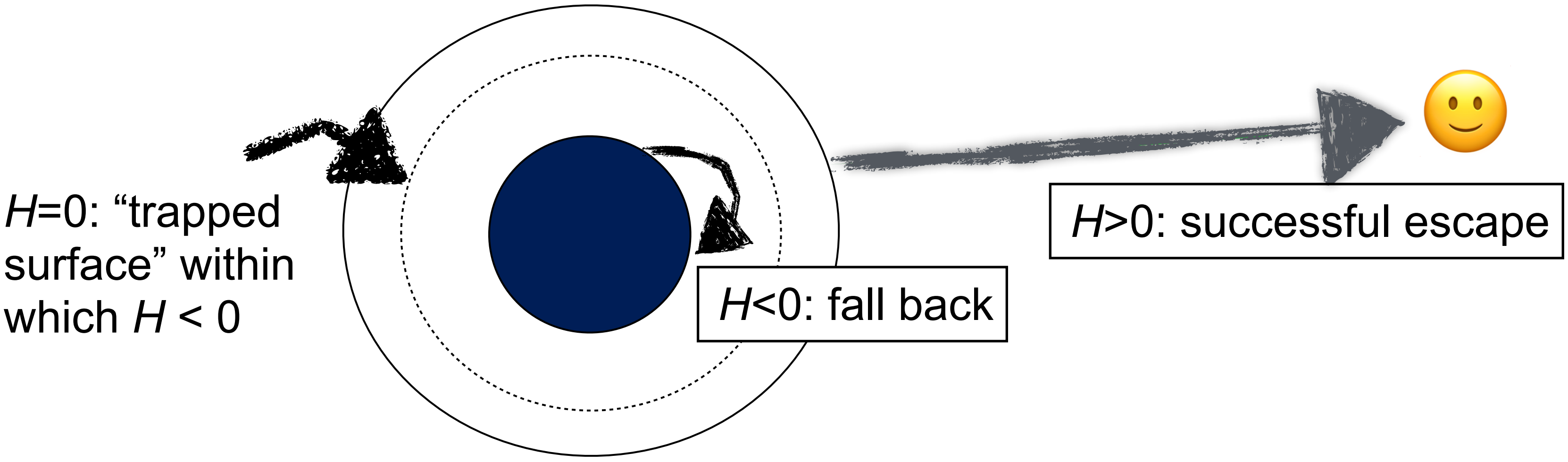
1967: John Wheeler’s “black hole”

1974: Stephen Hawking: black holes are grey, emitting thermal radiation

"Black objects"

$$H = 0: \quad \frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0$$

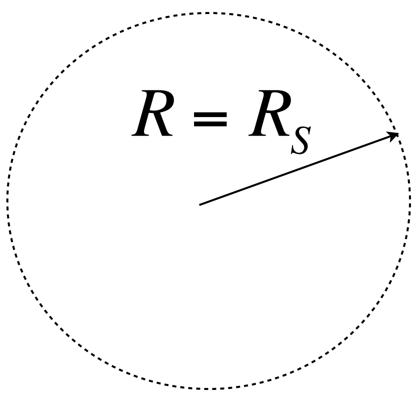
v_e : initial velocity at the surface ("escape velocity")



Black holes

Consider the limit $v_e = c$ (velocity of light)

$$\frac{1}{2}mc^2 - \frac{GMm}{R_S} = 0: \qquad R_S = \frac{2GM}{c^2}$$



Radius of a

Schwarzschild black hole

When does Newton's law of gravity apply?

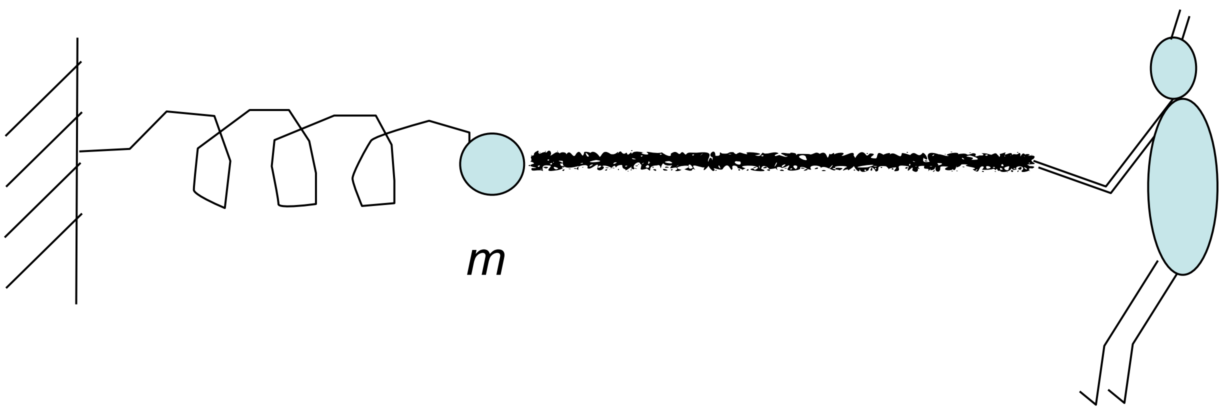
Newton's law of gravitational attraction to a mass *M* applies, provided that

a) **Distances** >> Schwarzschild radius of *M*

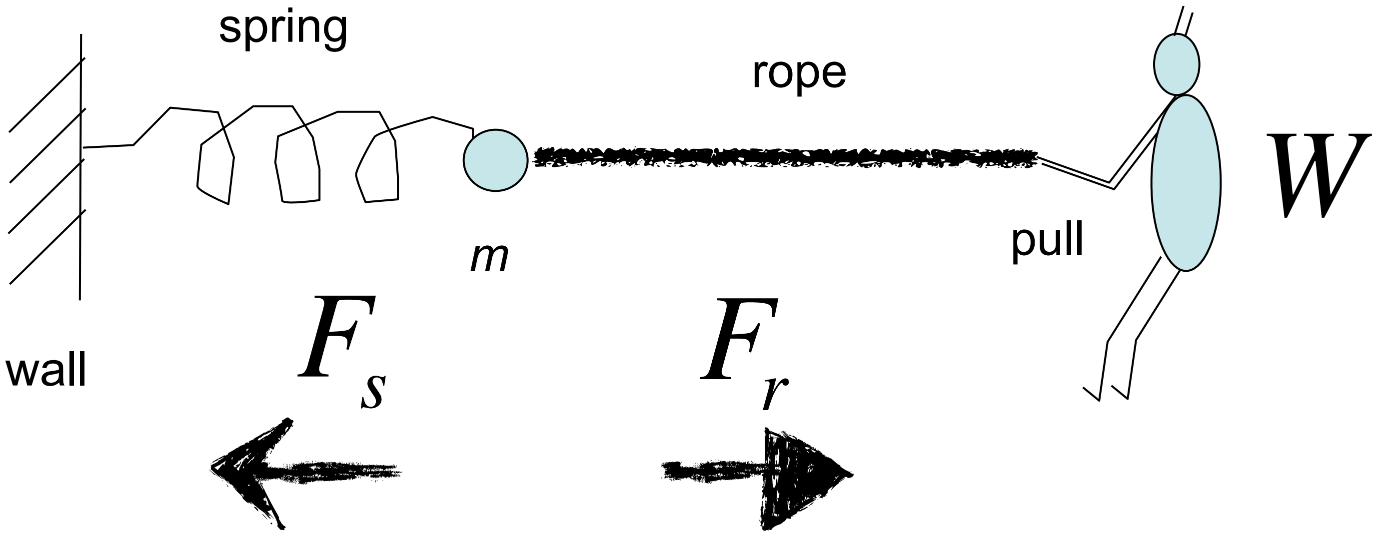
b) **Accelerations** >> cosmological background acceleration (defined by the velocity of light and the Hubble parameter)

Newton's law applies very well to the solar system.
Small deviations in orbit of Mercury. (Small but important!)

Force and energy



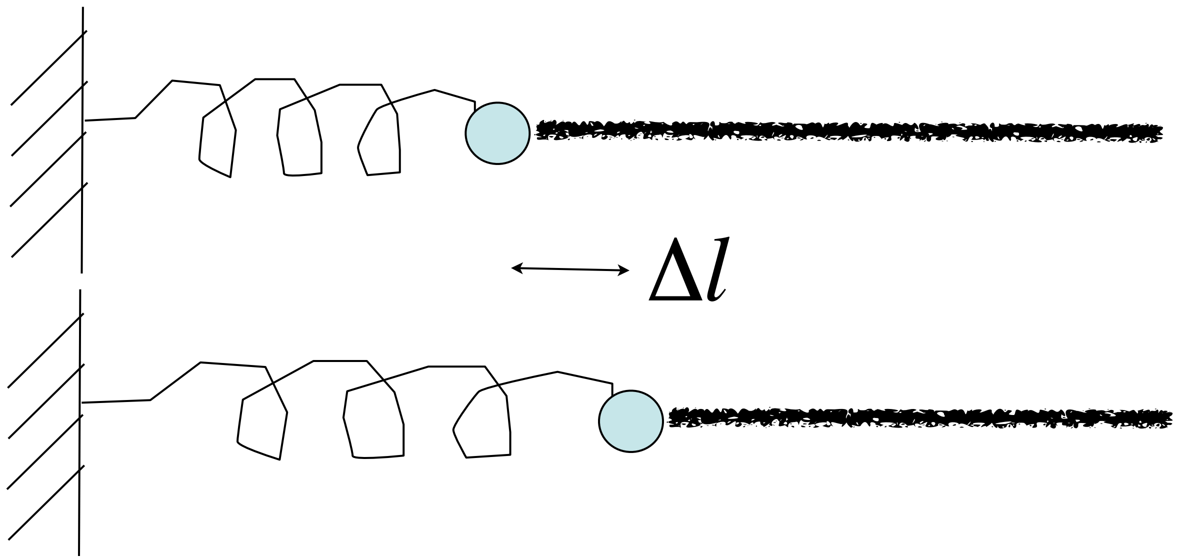
Work and potential energy



a) $$F_s = -F_r$$ Newton's third law

b) $$\Delta E_s = W$$ Work performed stored in spring potential energy

Work and potential energy



$$\Delta E_s = -\int_0^{\Delta l} F_s ds$$

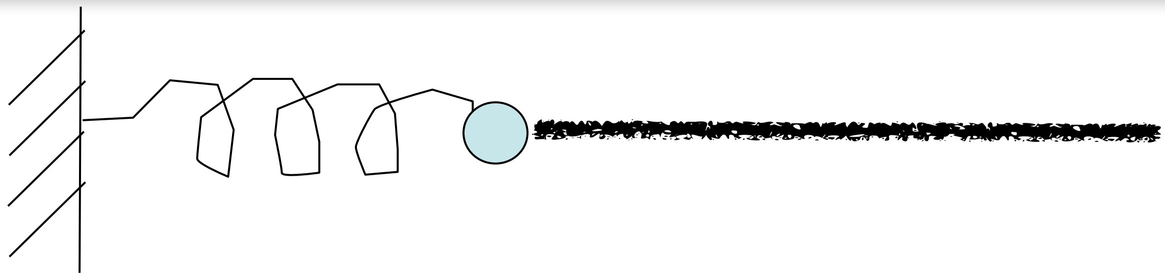
Newton's third law

$$\Delta E_s = W$$

$$W = \int_0^{\Delta l} F_r ds$$

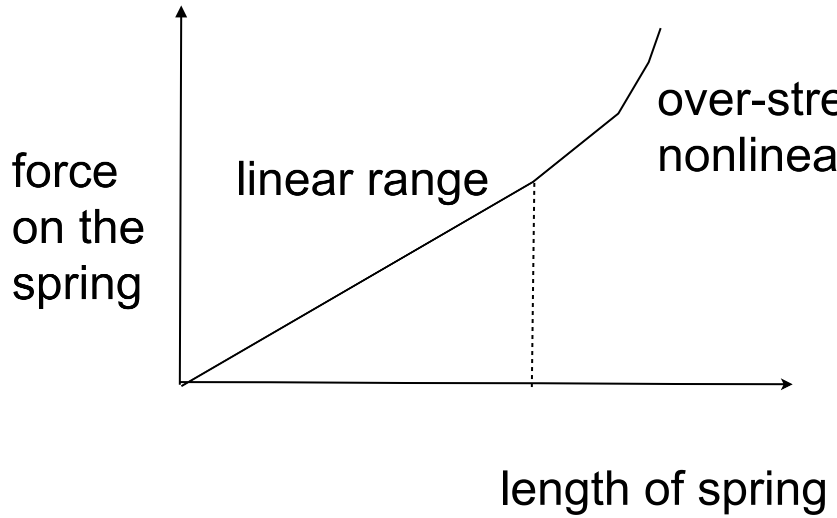
Changes are assumed to be slow and conservative, neglecting kinetic energy in *m* and dissipation by friction

– example: linear spring



Hooke's law (1660): $F_s = -kl$

k is spring constant: $[k] = \dfrac{[F]}{[l]} = \dfrac{\mathrm{g\,cm\,s^{-2}}}{\mathrm{cm}} = \mathrm{g\,s^{-2}}$



— Aside: Young's modulus

Formulation of Hooke's law in **dimensionless strain $\Delta L / L$**

— strain-stress correlation

Young: linear relationship between **stress** and **strain**

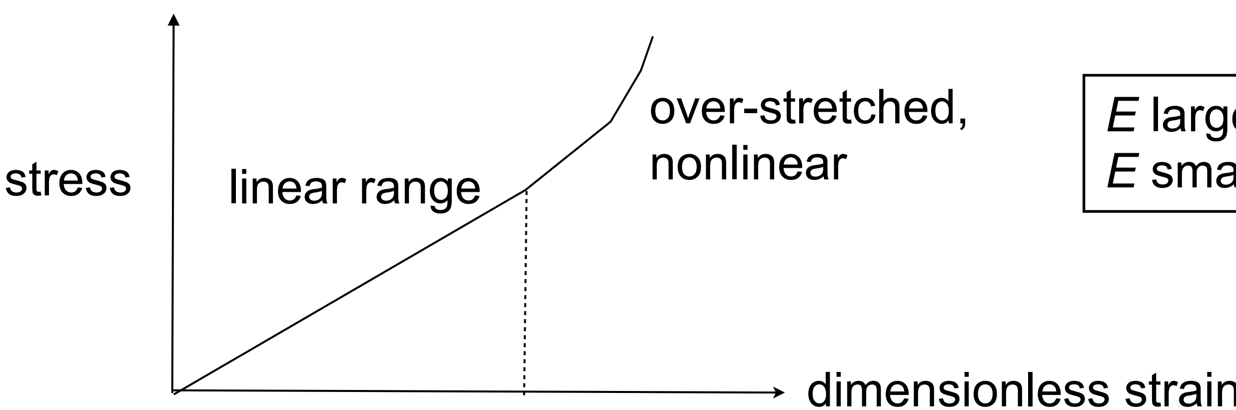
$$F = A\sigma$$

$$\sigma = E\frac{\Delta l}{l}$$

E is Young's modulus

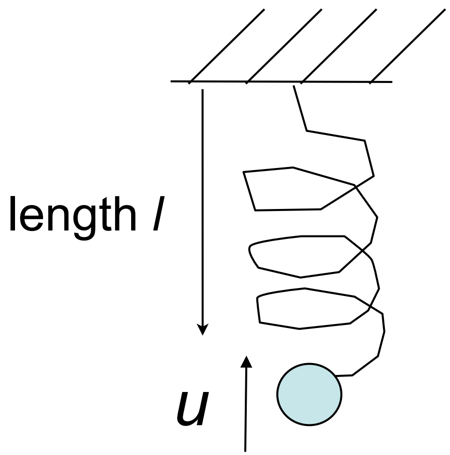
— strain-stress correlation

Dimensional analysis: $$[E] = \frac{[F/A]}{[\Delta l / l]} = \mathrm{g\,cm^{-1}s^{-2}}$$



E large: stiff material
E small: soft material

Hooke's pendulum clock

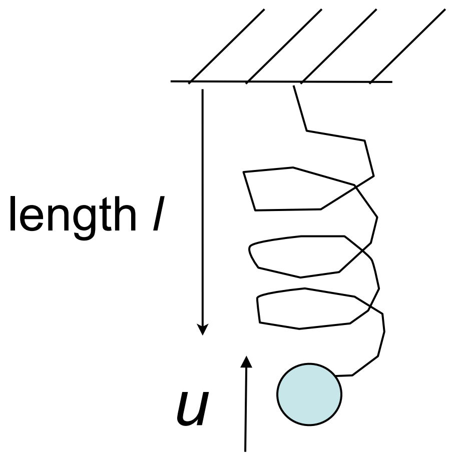


Hooke: linear relationship between **force** and **stretch**

$$F = ku$$

$$u = l_0 - l$$

Hooke's pendulum clock



Gravitational force balanced by a stretch of the spring:

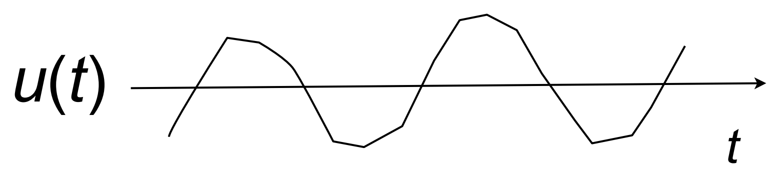
$$F_0 = mg = kl_0$$

$$l = l_0 - u$$

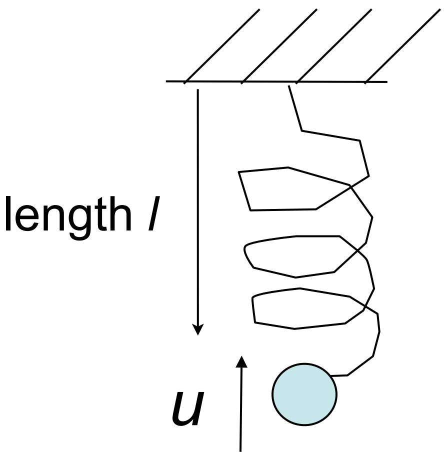
$$\Delta F = F - F_0 = -k(l - l_0) = -ku$$

Newton's third law $\Delta F = ma = m\ddot{u}$

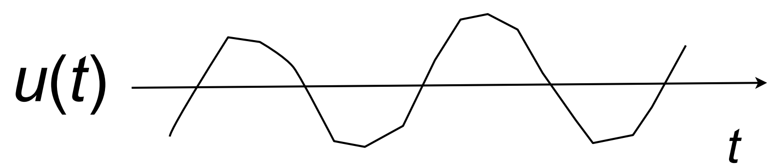
$$\boxed{m\ddot{u} = -ku}$$



Hooke's pendulum clock



Time-harmonic deflection



Equation of motion $\boxed{m\ddot{u} = -ku}$

is a 2nd order ordinary differential equation:

$$u = A\cos\left(\omega t + \varphi_0\right),$$

$$\omega = \sqrt{\frac{k}{m}},$$

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Two integration constants: amplitude A, initial phase φ_0

The free falling apple

Newton's description of particle motion

Newton's law of gravitation gives a precise description of particle motion and gravitation, culminating in Kepler's laws.

Newton uses his own differential and integral calculus:

1) Coordinates (polar or Cartesian) of particles in motion, for apples and moon's alike

2) Forces by contact or gravitation

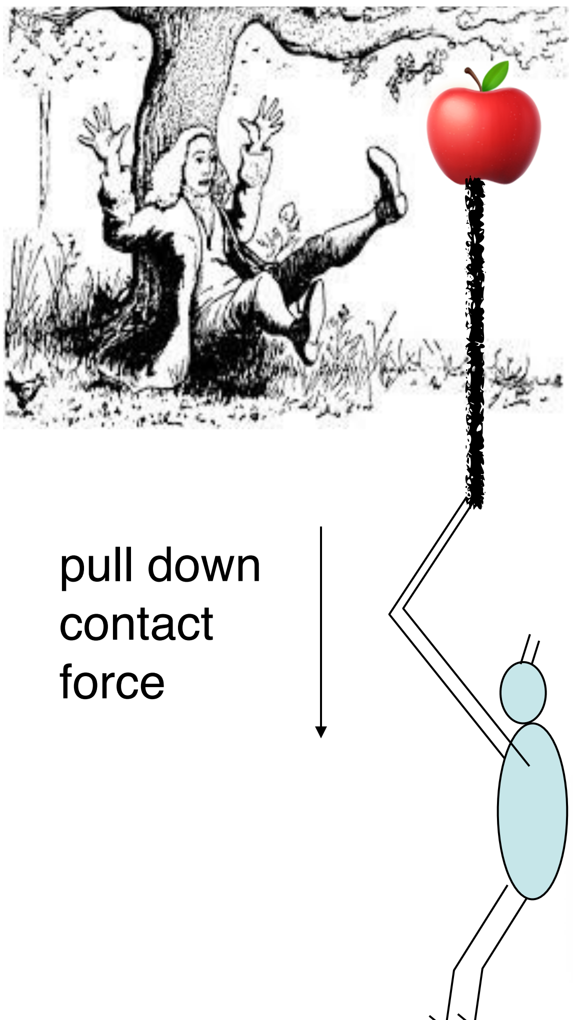
3) Momentum vectors and tangent vectors

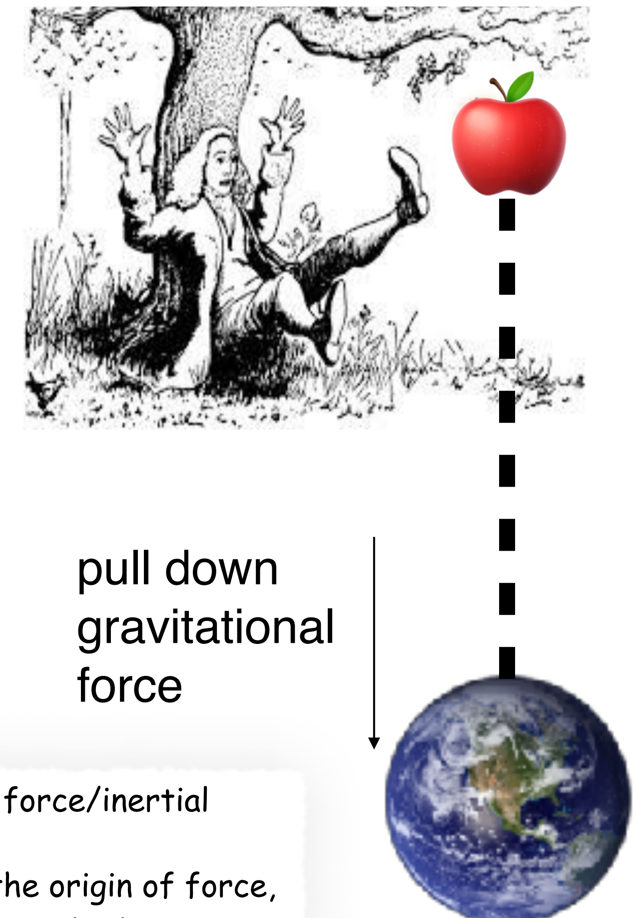
4) Trajectories as integrals of tangent vectors

5) Total energy and angular momentum as integrals of motion

Newton's equivalence principle

Force = mass x acceleration

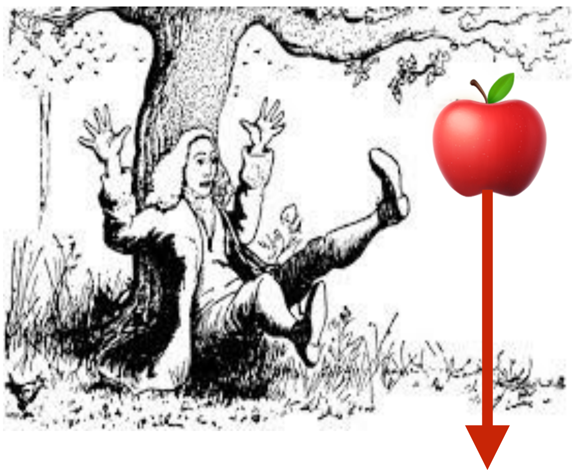


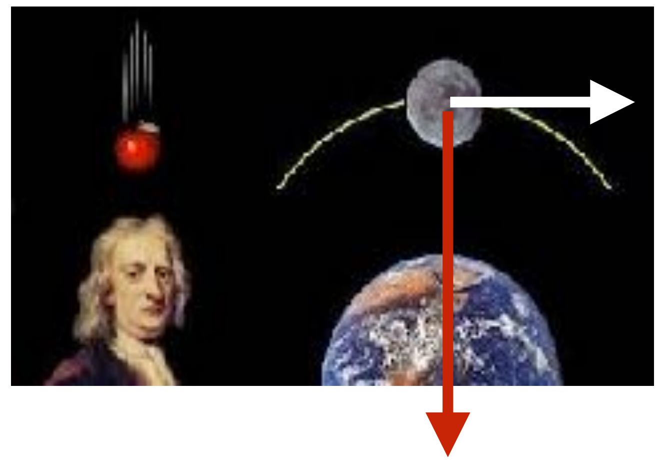


Acceleration = force/inertial mass,
regardless of the origin of force,
whether it be a contact or gravitational force, or otherwise

Momentum and time rate-of-change

momentum = mass x velocity





Direction of conserved momentum

direction of time rate-of-change of momentum

Momentum is conserved along direction of vanishing forces

Force = time rate-of-change of momentum

Vector calculus of position, velocity, momentum

$$\bar{x} = \begin{pmatrix} x \text{ coordinate position} \\ y \text{ coordinate position} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\bar{v} = \begin{pmatrix} x \text{ velocity} \\ y \text{ velocity} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\bar{p} = m\bar{v} = \begin{pmatrix} x \text{ momentum} \\ y \text{ momentum} \end{pmatrix} = m\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Example: force inferred from trajectory

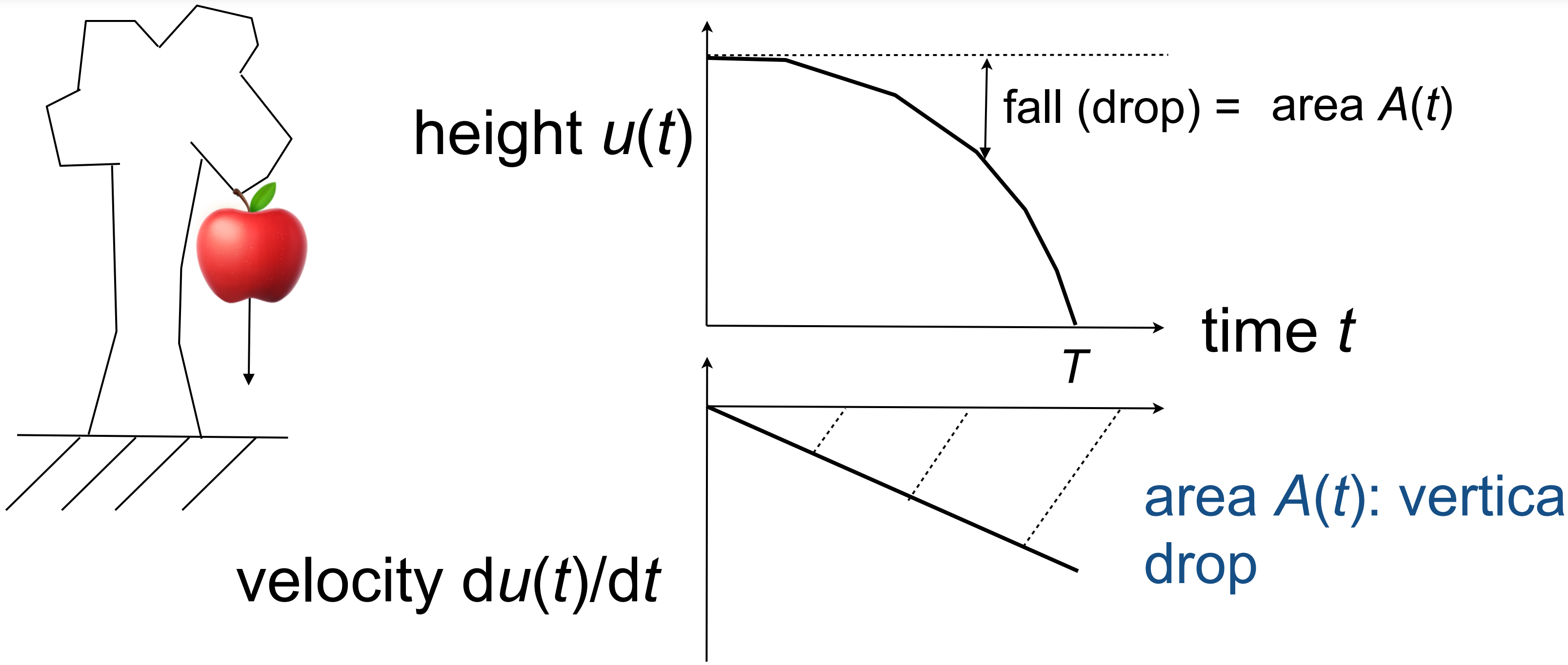
$$\bar{x}(t) = \begin{pmatrix} a - bt \\ c - dt^2 \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(a - bt) \\ \frac{d}{dt}(c - dt^2) \end{pmatrix} = -\begin{pmatrix} b \\ 2dt \end{pmatrix}$$

$$\bar{p} = m\bar{v} = -m\begin{pmatrix} b \\ 2dt \end{pmatrix}$$

$$\bar{F} \equiv \frac{d}{dt}\bar{p} = -m\begin{pmatrix} \frac{d}{dt}(b) \\ \frac{d}{dt}(2dt) \end{pmatrix} = -m\begin{pmatrix} 0 \\ 2d \end{pmatrix}$$

Question: is this force a contact force or a gravitational force?

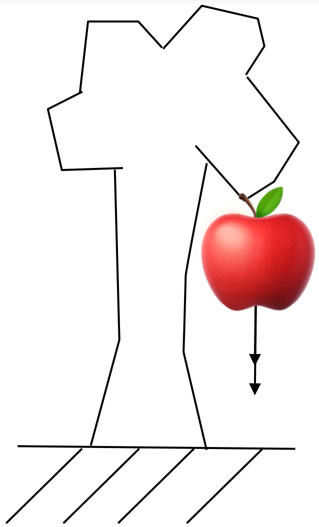
Example: Newton's apple from the tree



Apple's free fall Initial Value Problem (IVP):

$$\begin{cases} m\ddot{u}(t) = -mg \\ u(0) = H \\ \dot{u}(0) = 0 \end{cases}$$

— free fall time



Integrate:

$$\dot{u}(t) - \dot{u}(0) = -gt: \quad \dot{u}(t) = -gt$$

Integrate a second time:

$$u(t) - u(0) = -\frac{1}{2}gt^2: \quad u(t) = H - \frac{1}{2}gt^2$$

$u(t) = H - A$ with vertical drop $A(t) = \frac{1}{2} g\, t^2$

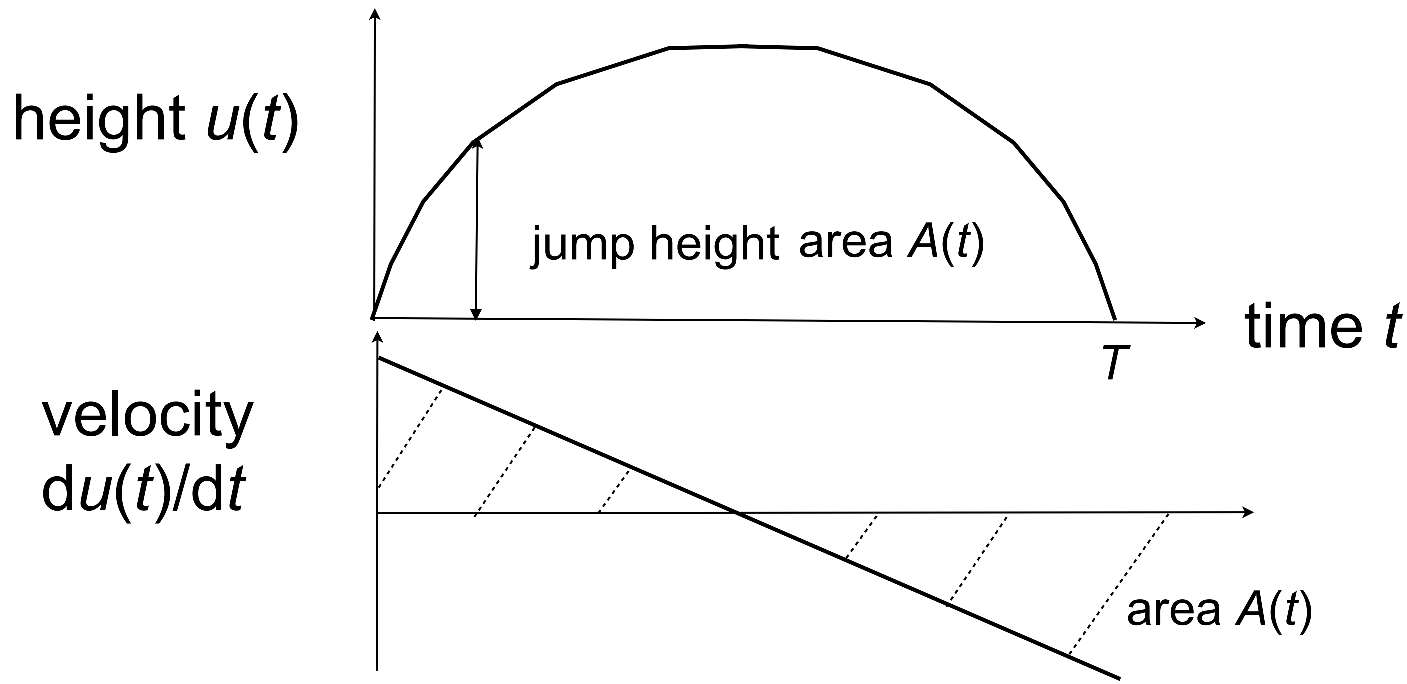
Free fall time:

$$u(T) = 0: \quad T = \sqrt{\frac{2H}{g}}$$

The jumper

http://www.wired.com/2014/04/basketball-physics/

Jumper trajectory in coordinate space



Jumper's IVP:

$$\begin{cases} m\ddot{u}(t) = -mg \\ u(0) = 0 \\ \dot{u}(0) = V \end{cases}$$

— total flight time

Integrate once:

$$\dot{u}(t) - \dot{u}(0) = -gt: \qquad \dot{u}(t) = V - gt$$

Integrate a second time:

$$u(t) - u(0) = Vt - \frac{1}{2}gt^2: \qquad u(t) = Vt - \frac{1}{2}gt^2$$

Time at maximal height $\quad \dot{u}(t) = 0: \quad t_* = \dfrac{V}{g}$

Total flight time $\quad u(t) = 0: \quad T = \dfrac{2V}{g}$

— jump height

Parabolic trajectory

$$u(t) = Vt - \frac{1}{2}gt^2 = VT\ y(\tau), \quad y(\tau) = \tau(1-\tau)$$

in dimensionless time: $\tau = \dfrac{t}{T}$

Parabolic curves: $\max y(\tau) = \dfrac{1}{4} \quad \left(\tau = \dfrac{1}{2}\right)$

Maximal height reached:

$$\boxed{\frac{1}{4}VT = \frac{V^2}{2g}} = \frac{E_k}{mg}$$

- turning point

$$H = E_k + U$$

Conserved total energy

$$E_k = \frac{1}{2} m v^2$$

Kinetic energy, maximal when potential energy is minimal

$$U = mgz$$

Potential energy, maximal when kinetic energy is minimal (zero)

$$H = \frac{1}{2} m v^2 + mgz = \begin{cases} \frac{1}{2} m V^2 & \text{jumper on the ground} \\ mgh & \text{jumper at maximal height } h \end{cases}$$

- maximal height at turning point

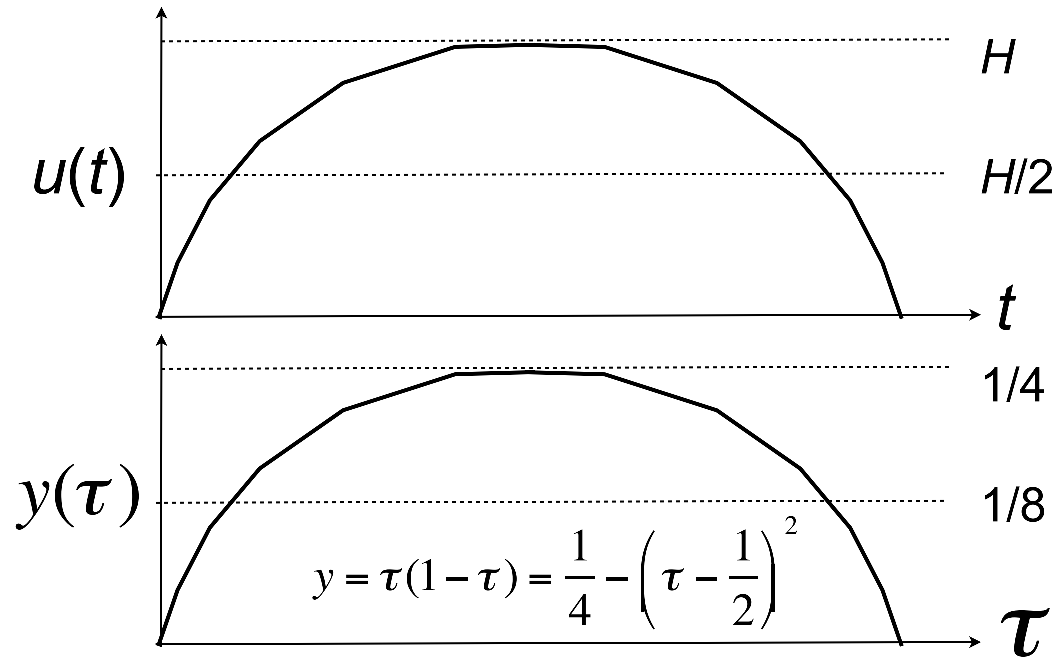
$$H(0) = H(h):$$

$$\frac{1}{2}mV^2 + 0 = E_k + U\Big|_{z=0} = E_k + U\Big|_{z=h} = 0 + mgh$$

$$mgh = \frac{1}{2}mV^2: \quad h = \frac{\frac{1}{2}mV^2}{mg} = \frac{V^2}{2mg}$$

once more

— duration of flying high



Solve $y(\tau)=\frac{1}{8}$

$$\frac{1}{8}=\left(\tau-\frac{1}{2}\right)^2 : \quad \tau_{\pm}=\frac{1}{2}\pm\frac{1}{2\sqrt{2}}$$

Time of flight above one-half the maximum height is

$$\frac{t_+-t_-}{T}=\tau_+-\tau_-=\frac{1}{\sqrt{2}}\cong 0.71$$

In-class Assignment

Position vector $$\overline{p}(t) = \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix} = t\,\boldsymbol{i} + \sqrt{1-t^2}\,\boldsymbol{j} \quad (-1 < t < 1)$$

Tangent vector $$\overline{\tau}(t) = \frac{d\overline{p}(t)}{dt}$$

1. Plot the trajectory in the two-dimensional plane (x,y), as t traverses from -1 to 1.

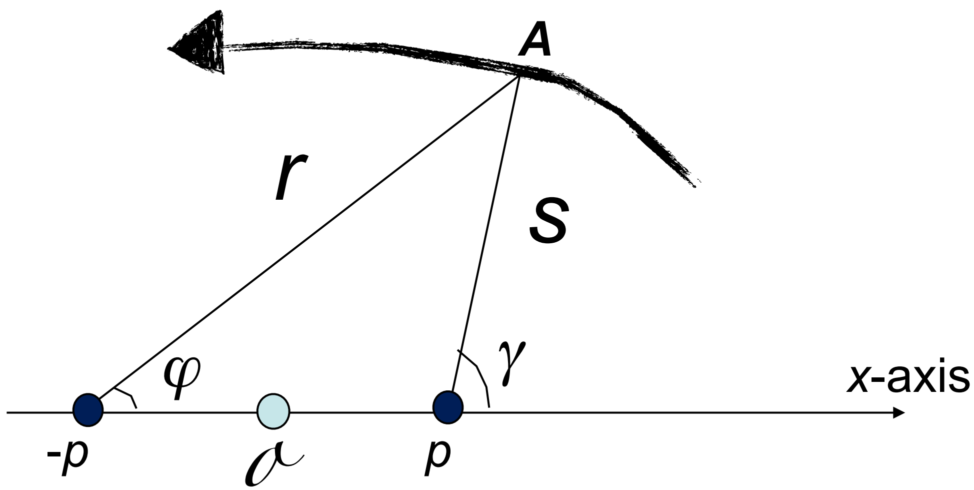
2. Compute the tangent vector, and sketch the tangent vector to the trajectory for t=-1,0,1 in your plot.

3. Perform the substitution $t = \cos\varphi \quad (-\pi < \varphi < \pi)$

Give explicit expressions for $\overline{p}(\varphi), \quad \overline{\tau}(\varphi)$

4. What can you say about the angle between the position and tangent vector?

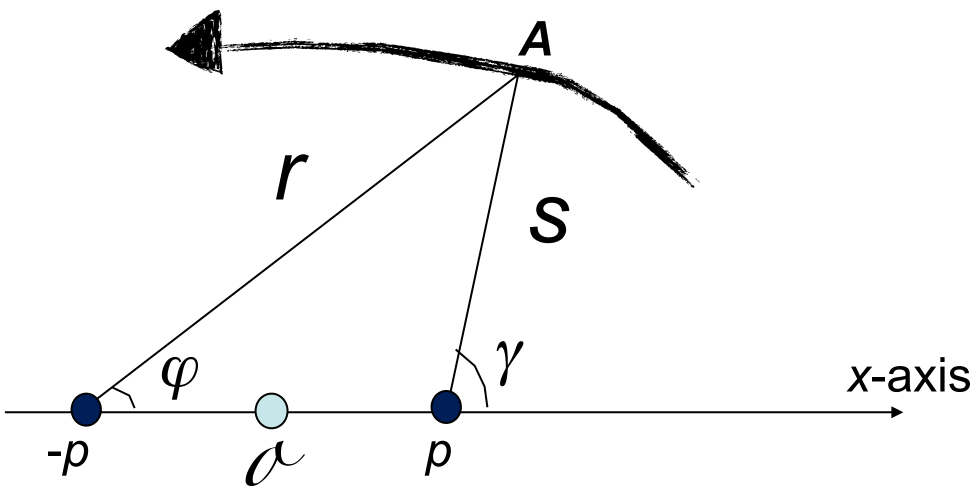
Aside: Kepler's orbits in polar coordinates



$$\begin{cases} r\sin\varphi = s\sin\gamma \\ r\cos\varphi = 2p + l\cos\gamma \end{cases}$$

A's trajectory is an ellipse: $r + s = c$ (c is some constant)

— polar coordinates

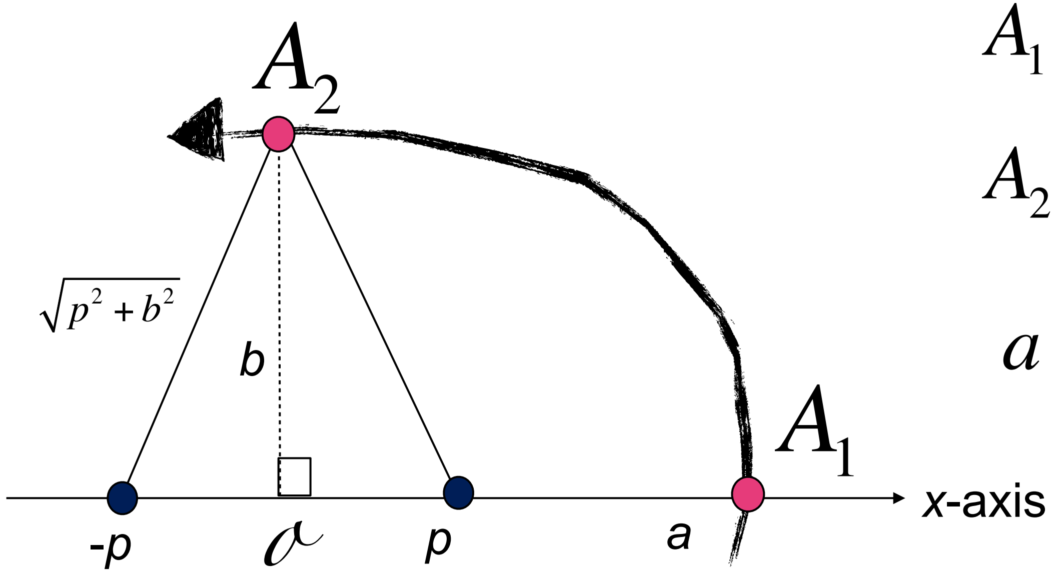


Write out:

$$\begin{cases} r^2 = s^2 + 4ps\cos\gamma + 4p^2 \\ s\cos\gamma = r\cos\varphi - 2p \\ s = c - r \end{cases}$$

$$\boxed{r(\varphi) = \frac{\dfrac{1}{2}c - \dfrac{2p^2}{c}}{1 - \dfrac{2p^2}{c}\cos\varphi}}$$

— ellipticity



$$A_1: \quad 2(a+p)-2p=c$$

$$A_2: \quad 2\sqrt{p^2+b^2}=c$$

$$a=\sqrt{p^2+b^2}, c=2a$$

Ellipticity e: $\quad p \equiv ea$

$$\boxed{r(\varphi)=a\frac{1-e^2}{1-e\cos\varphi}}$$

In Newton's theory, *u*=1/*r* is harmonic in phi, unifying elliptic and hyperbolic orbits.